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(Research Article)

Point Collocation Method for the Analysis of Euler-Bernoulli Beam on Winkler Foundation

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Abstract

In this study, the point collocation method was implemented to solve the boundary value problem (BVP) of simply supported Euler-Bernoulli beams resting on Winkler foundation for uniformly distributed load. Mathematically, the BVP solved was a fourth order non homogeneous ordinary differential equation with constant parameters for the case of prismatic cross-sections considered. A two term deflection function was used to determine the residual function. Dirac delta functions at the collocation points were used in a weighted residual statement of the problem to obtain the collocation equations which were solved to obtain the unknown parameters. The results obtained were reasonably in agreement with the solutions obtained in literature using 3 term Ritz solutions. The difference between the 2 term collocation and the 3 term Ritz solutions were insignificant at less than 5.02% considering the obvious simplicity offered by the point collocation method.

Keywords: Point collocation method, Euler-Bernoulli beam, Winkler foundation, Residual (error) function, Dirac delta function, Boundary value problem.

1. Introduction

The concepts of beams resting on elastic foundations have been extensively applied by geotechnical, road pavement and railway engineers in the analysis and design of railroads, road pavements and foundations. It is also used in the analysis and design of buried gas pipeline systems [1-4]. The application entailed the formulation of theories of beams as well as theories for the elastic foundation. Several theories were proposed for beams. They are Bernoulli-Euler beam theory [4, 5], Timoshenko beam theory, Mindlin beam theory, refined shear deformation beam theory etc.

The Bernoulli-Euler beam theory which neglects the effect of shear deformation on the flexural behaviour of thin (slender) beams has been adopted as the beam model in this work. Several theories/models have also been proposed for the elastic foundation. They include: Winkler [6] model, Filonenko-Borodich model, Hetenyi model, Pasternak [7] model, Vlasov and Leontiev model, Kerr [8] model.

The Winkler model which is the simplest model assumes that the reaction forces of the soil on the beam are directly proportional at every point on the beam to the deflection of the beam at that point. The physical representation of the Winkler model is a bed of continuous closely spaced linear elastic springs which define the vertical deformation characteristics of the foundation bed. The constant of proportionality of the springs is defined as the modulus of subgrade reaction, *k*. The Winkler one parameter model representation of the foundation, though simple, does not accurately represent the characteristics of many practical foundations. A major defect of the Winkler foundation model is the displacement discontinuity that appears between the loaded and unloaded parts of the foundation surface, which violates the elasticity behaviour of the soil.

Other researchers have improved on the Winkler idealization by introducing shear interaction between the Winkler spring elements to take care of the displacement discontinuity in the Winkler model. Those models are generally called two parameter or three parameter foundation models.

Despite the inadequacies of the Winkler foundation model, it has found extensive application in soil structure interaction analysis due to its obvious simplicity. This study focuses on Euler-Bernoulli beam resting on Winkler foundation. The solution is a statically indeterminate problem of mechanics. Several techniques have been employed to solve the fourth order equation governing the Euler-Bernoulli beam on Winkler foundation problem. The solution methods are broadly classified as analytical and numerical solutions. The analytical solutions are [9]: integral transform methods,

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product method, and eigen function expansion method. Not all problems of Euler-Bernoulli beam on Winkler foundation can be solved by analytical methods since their solutions are very complicated.

Numerical methods for solving the Euler-Bernoulli beam on Winkler foundation problem include: finite element methods [10, 11], finite difference methods, Galerkin methods, Spline collocation methods [12, 13], isogeometric collocation method [5], interpolation methods [14] and differential transform methods [3].

2. Research aim and objectives

The general aim of this work is to use the point collocation method to solve the flexural problem of Euler-Bernoulli beam on Winkler foundation for the case of simply supported ends at $x = 0$, $x = l$, and uniformly distributed transverse load. The specific objectives are:

1) To find suitable approximating deflection functions for the problem in terms of deflection shape functions and unknown displacement parameters.
2) To find the residual function for the Euler-Bernoulli beam on Winkler foundation
3) To obtain a solution to the Euler-Bernoulli beam on Winkler foundation problem at discrete points on the beam called the collocation points.
4) To find the deflection and bending moment and the maximum values of the deflection and bending moment for given values of k/EI.
5) To compare the solutions obtained for maximum deflection and maximum bending moment with the solutions obtained using Ritz variational method.

3. Methodology

The governing ordinary differential equation (ODE) for Euler-Bernoulli beam on Winkler foundation is the fourth order equation:

$$EI\frac{d^4w(x)}{dx^4} + kw(x) = q(x) \tag{1}$$

where E is the Young's modulus of elasticity of the beam material, I is the moment of inertia, k is the modulus of soil reaction or the Winkler modulus, $q(x)$ is the applied transverse distributed load on the beam and $w(x)$ is the deflection of the beam. x is the longitudinal axis of the beam. Let

$$\frac{k}{EI} = 4\lambda^4 \tag{2}$$

Then the governing ODE becomes:

$$\frac{d^4w}{dx^4} + 4\lambda^4 w = \frac{q}{EI} \tag{3}$$

The work considers an Euler-Bernoulli beam of prismatic cross-section with simply supported ends at $x = 0$, and $x = l$, where l is the length of the beam.

The boundary conditions become:

$$w(x=0) = w(x=l) = 0 \tag{4}$$

And

$$M(x=0) = M(x=l) = 0 \tag{5}$$

Where M is the bending moment distribution.

Using the bending moment-displacement relations for Euler-Bernoulli beams the boundary conditions Equation (5) could be expressed as:

$$w''(x=0) = w''(x=l) = 0 \tag{6}$$

Where the primes denote differentiation with respect to the x-coordinate.

A suitable displacement coordinate function that satisfies the boundary conditions of simple supports at $x = 0$, and $x = l$ is:

$$w(x) = c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{3\pi x}{l} \tag{7}$$

where $\sin \pi x/l$ and $\sin 3\pi x/l$ are the displacement shape (basis) functions, and c_1 and c_2 are the unknown displacement parameters of the basis functions.

The residual function $R(x)$ is obtained as:

$$R(x) = c_1\left(\frac{\pi}{l}\right)^4 \sin\frac{\pi x}{l} + c_2\frac{81\pi^4}{l^4}\sin\frac{3\pi x}{l} + 4\lambda^4 c_1 \sin\frac{\pi x}{l} + 4\lambda^4 c_2 \sin\frac{3\pi x}{l} - \frac{q}{EI} \tag{8}$$

Simplifying,

$$R(x) = \left(\frac{\pi^4}{l^4} + 4\lambda^4\right)c_1 \sin\frac{\pi x}{l} + \left(\frac{81\pi^4}{l^4} + 4\lambda^4\right)c_2 \sin\frac{3\pi x}{l} - \frac{q}{EI} \tag{9}$$

4. Results

The collocation points are chosen at $x = l/4$ and $x = l/2$, and the collocation equations are obtained using the Dirac delta functions at collocation points as the weighting functions, in a weighted residual integral statement of the problem to obtain the following two equations:

$$\int_0^l R(x)\delta(x=l/4)dx=0 \quad (10)$$

$$\int_0^l R(x)\delta(x=l/2)dx=0 \quad (11)$$

The collocation equations become:

For $x = l/4$,

$$R\left(x=\tfrac{l}{4}\right)=\left(\frac{\pi^4}{l^4}+4\lambda^4\right)c_1\frac{\sqrt{2}}{2}+\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)c_2\frac{\sqrt{2}}{2}-\frac{q}{EI}=0 \quad \ldots (12)$$

And for $x = l/2$,

$$R\left(x=\tfrac{l}{2}\right)=\left(\frac{\pi^4}{l^4}+4\lambda^4\right)c_1+\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)c_2(-1)-\frac{q}{EI}=0 \quad \ldots (13)$$

Alternatively,

$$\left(\frac{\pi^4}{l^4}+4\lambda^4\right)c_1-\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)c_2=\frac{q}{EI} \quad (14)$$

$$\left(\frac{\pi^4}{l^4}+4\lambda^4\right)c_1+\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)c_2=\frac{2}{\sqrt{2}}\frac{q}{EI} \quad (15)$$

Solving,

$$2\left(\frac{\pi^4}{l^4}+4\lambda^4\right)c_1=\frac{q}{EI}\left(1+\frac{2}{\sqrt{2}}\right)=\frac{q}{EI}\left(\frac{\sqrt{2}+2}{\sqrt{2}}\right) \quad (16)$$

$$c_1=\frac{q}{EI}\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\cdot\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)} \quad (17)$$

Similarly, solving for c_2 in Equations (14) and (15), we obtain:

$$-2\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)c_2=\frac{q}{EI}-\frac{q}{EI}\frac{2}{\sqrt{2}}=\frac{q}{EI}\left(1-\frac{2}{\sqrt{2}}\right)=\frac{q}{EI}\left(\frac{\sqrt{2}-2}{\sqrt{2}}\right) \quad \ldots (18)$$

$$c_2=\frac{q}{EI}\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)\cdot\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)} \quad (19)$$

The deflection is then found as:

$$w(x)=\frac{q}{EI}\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}\sin\frac{\pi x}{l}+\frac{q}{EI}\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\sin\frac{3\pi x}{l} \quad (20)$$

The center deflection w_c, is found from:

$w\left(x=\tfrac{l}{2}\right)=c_1-c_2$, as

$$w_c=\frac{q}{EI}\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}-\frac{q}{EI}\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)} \quad \ldots (21)$$

$$w_c=\frac{q}{2\sqrt{2}EI}\left(\frac{2+\sqrt{2}}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}-\frac{2-\sqrt{2}}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\right) \quad (22)$$

The bending moment distribution $M(x)$ is found from the bending moment-deflection relation:

$$M(x)=-EI\,w''(x) \quad (23)$$

Thus,

$$M(x)=EI\left(c_1\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}+\frac{9\pi^2}{l^2}c_2\sin\frac{3\pi x}{l}\right) \quad (24)$$

$$M(x)=EI\frac{\pi^2}{l^2}\left(c_1\sin\frac{\pi x}{l}+9c_2\sin\frac{3\pi x}{l}\right) \quad (25)$$

$$M(x)=\frac{q\pi^2}{l^2}\left(\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}\sin\frac{\pi x}{l}+\frac{9(2-\sqrt{2})}{2\sqrt{2}}\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\sin\frac{3\pi x}{l}\right) \quad (26)$$

At the center, $x = l/2$, the bending moment at the center, M_c, becomes:

$$M_c=M\left(x=\tfrac{l}{2}\right) \quad (27)$$

$$M_c=\frac{q\pi^2}{l^2}\left(\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}\sin\frac{\pi}{2}\right.$$

$$+\frac{9(2-\sqrt{2})}{2\sqrt{2}}\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\sin\frac{3\pi}{2}\Bigg) \quad (28)$$

$$M_c=\frac{q\pi^2}{2\sqrt{2}\,l^2}\left(\frac{2+\sqrt{2}}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}-\frac{9(2-\sqrt{2})}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\right) \quad (29)$$

The shear force distribution $Q(x)$ is found from the shear force-deflection relation as:

$$Q(x)=-EIw'''(x)=EI\left(c_1\frac{\pi^3}{l^3}\cos\frac{\pi x}{l}+\frac{27\pi^3}{l^3}c_2\cos\frac{3\pi x}{l}\right) \quad \ldots (30)$$

$$Q(x)=\frac{EI\pi^3}{l^3}\left(c_1\cos\frac{\pi x}{l}+27c_2\cos\frac{3\pi x}{l}\right) \quad (31)$$

$$Q(x)=\frac{EI\pi^3}{l^3}\Bigg(\frac{q}{EI}\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}\cos\frac{\pi x}{l}$$
$$+\frac{27q}{EI}\left(\frac{2-\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\cos\frac{3\pi x}{l}\Bigg) \quad (32)$$

$$Q(x)=\frac{q\pi^3}{l^3}\Bigg(\left(\frac{2+\sqrt{2}}{2\sqrt{2}}\right)\frac{1}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}\cos\frac{\pi x}{l}$$
$$+\frac{27(2-\sqrt{2})}{2\sqrt{2}}\frac{1}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\cos\frac{3\pi x}{l}\Bigg) \quad (33)$$

For $x = 0$,

$$Q(0)=\frac{q\pi^3}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\left(\frac{\pi^4}{l^4}+4\lambda^4\right)}+\frac{27(2-\sqrt{2})}{\left(\frac{81\pi^4}{l^4}+4\lambda^4\right)}\right) \quad (34)$$

$$Q(x=l)=-Q(0) \quad (35)$$

Numerical solutions

$$\text{For } \frac{kl^4}{EI}=4, \quad (36)$$

$$\frac{k}{EI}=\frac{4}{l^4}=4\lambda^4 \quad (37)$$

$$w_c=\frac{q}{2\sqrt{2}\,EI}\left(\frac{2+\sqrt{2}}{\frac{\pi^4}{l^4}+\frac{4}{l^4}}-\frac{2-\sqrt{2}}{\frac{81\pi^4}{l^4}+\frac{4}{l^4}}\right) \quad (38)$$

$$w_c=0.011877055\frac{ql^4}{EI} \quad (39)$$

$$M_c=\frac{q\pi^2}{2\sqrt{2}\,l^2}\left(\frac{2+\sqrt{2}}{\frac{\pi^4}{l^4}+\frac{4}{l^4}}-\frac{9(2-\sqrt{2})}{\frac{81\pi^4}{l^4}+\frac{4}{l^4}}\right) \quad (40)$$

$$M_c=0.11515ql^2 \quad (41)$$

$$Q(0)=\frac{q\pi^4}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\frac{\pi^4}{l^4}+\frac{4}{l^4}}+\frac{27(2-\sqrt{2})}{\frac{81\pi^4}{l^4}+\frac{4}{l^4}}\right) \quad (42)$$

$$Q(0)=\frac{q\pi^4}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\frac{4+\pi^4}{l^4}}+\frac{27(2-\sqrt{2})}{\frac{81\pi^4+4}{l^4}}\right) \quad (43)$$

$$Q(0)=1.22848936ql=-Q(l) \quad (44)$$

$$\text{For } \frac{kl^4}{EI}=10, \quad (45)$$

$$4\lambda^4=\frac{k}{EI}=\frac{10}{l^4}, \quad (46)$$

$$w_c=\frac{q}{2\sqrt{2}\,EI}\left(\frac{2+\sqrt{2}}{\frac{\pi^4}{l^4}+\frac{10}{l^4}}-\frac{2-\sqrt{2}}{\frac{81\pi^4}{l^4}+\frac{10}{l^4}}\right) \quad (47)$$

$$w_c=\frac{ql^4}{2\sqrt{2}\,EI}\left(\frac{2+\sqrt{2}}{\pi^4+10}-\frac{2-\sqrt{2}}{81\pi^4+10}\right) \quad (48)$$

$$w_c=0.01121219\frac{ql^4}{EI} \quad (49)$$

$$M_c=\frac{q\pi^2}{2\sqrt{2}\,l^2}\left(\frac{2+\sqrt{2}}{\frac{\pi^4+10}{l^4}}-\frac{9(2-\sqrt{2})}{\frac{81\pi^4+10}{l^4}}\right) \quad (50)$$

$$M_c\approx 0.10859ql^2 \quad (51)$$

The 3-term Ritz solution for center deflection w_c is:

$$w_c=0.011804482\frac{ql^4}{EI} \quad (52)$$

While the 3-term Ritz solution for center bending moment is:

$$M_c = 0.113255ql^2 \tag{53}$$

$$Q(0) = \frac{q\pi^4}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\frac{\pi^4}{l^4}+\frac{10}{l^4}} + \frac{27(2-\sqrt{2})}{\frac{81\pi^4}{l^4}+\frac{10}{l^4}}\right) \tag{54}$$

$$Q(0) \approx 1.16367ql \tag{55}$$

5. Discussion

In this study, the point collocation method was successfully implemented to solve the boundary value problem (BVP) of a prismatic Euler-Bernoulli beam resting on Winkler foundation, for the case of uniformly distributed transverse load and simply supported edges. The BVP was represented by a fourth order ordinary differential equation given by Equation (1), and for prismatic cross-section and homogeneous foundation, the equation has constant coefficients. A suitable deflection function with two unknown deflection parameters, c_1, and c_2, that satisfies the simply supported boundary conditions at $x = 0$, and $x = l$ was constructed from the deflection shape function of simply supported beams, as Equation (7). The residual (error) function was found in terms of the two unknown deflection parameters, as Equation (9). The collocation equations were written with the Dirac delta functions at the two collocation points ($x = l/4$ and $x = l/2$) used as the weighting functions. This yielded the two collocation Equations (12) and (13) in terms of the unknown deflection parameters. The collocation equations were expressed in matrix format as Equations (14) and (15). The collocation equations were solved using the techniques of linear algebra to obtain the unknown deflection parameters as Equation (17) and (19). The deflection was thus completely determined as Equation (20). The deflection at the center of the Euler-Bernoulli beam on Winkler foundation was found in general as Equation (22). The bending moment-deflection relation for Euler-Bernoulli beams theory (Equation (23)) was used to obtain the bending moment distribution along the longitudinal axis of the beam as Equation (26). The bending moment at the center was found as Equation (29). The shear force distribution along the longitudinal axis of the beam was obtained as Equation (33). Maximum values of shear force were found to occur at the ends ($x = 0$, $x = l$) of the beam and were obtained as Equations (34) and (35). Numerical solutions were obtained for the general solutions presented for values of the dimensionless Winkler parameter K, $\left(K = \frac{kl^4}{EI}\right)$ given by $K = 4$, and $K = 10$, where $K = 4\lambda^4 l^4$. For $K = 4$, the center deflection was obtained as Equation (39), while the center bending moment was obtained as Equation (41). The shear force at the beam end was obtained as Equation (44). For $K = 10$, the center deflection was obtained as Equation (49); the center bending moment was found as Equation (51). The shear force at the end was obtained as Equation (55). A comparison of the values of the center deflection for $K = 4$, and $K = 10$, shows that the center deflection reduced with increase in the Winkler modulus k. Similarly, comparison of the values of the center bending moments for $K = 4$ and $K = 10$, shows that the center bending moment reduced with increase in the Winkler modulus, k. The shear force at the ends increased with increase in the Winkler modulus. A comparison of the two term point collocation solution for center deflection with the three term Ritz solution showed a relative difference of –5.02% for the case of the non-dimensional Winkler modulus $K = 10$. The two term point collocation solution for center bending moments showed a relative difference of –4.12% when compared with the three term Ritz solution for center bending moment for the case of dimensionless Winkler modulus $K = 10$. Detailed calculation steps for the solution of deflection, shear force and bending moment at the centre and the ends ($x = 0$, $x = l$) for the various values of the dimensionless Winkler parameter are presented in Appendices 1, 2, 3, 4 and 5.

6. Conclusions

From the study, the following conclusions are made:

1. The point collocation method for solving BVP has been successfully applied to the flexural problem of simply supported Euler-Bernoulli beam resting on Winkler foundation.
2. The point collocation method relies on finding deflection function that satisfies the boundary conditions of deformation at the supports and the force boundary conditions (natural and essential boundary conditions).
3. The collocation equations are setup from the residual (error) functions by using the Dirac delta functions at the defined collocation points as the weighting functions in a weighted variational statement of problem.
4. The unknown deflection parameters are found using linear algebra techniques.
5. Maximum deflections and maximum bending moments occur at the center of the beam's longitudinal axis; and this is expected from considerations of symmetry of the beam and symmetry of loading.
6. Maximum shear force occurs at the simple supports (at $x = 0$, $x = l$).
7. The deflection at the center of the beam's longitudinal axis reduces as the Winkler modulus, k increases.
8. The bending moment at the center of the beam's longitudinal axis reduces as the Winkler modulus, k increases.
9. The shear force at the ends of the beam increase with increase in the Winkler modulus, k.

Nomenclature

E	Young's modulus of elasticity
I	moment of inertia
k	modulus of soil reaction
$q(x)$	applied transverse load distribution on the beam
x	longitudinal axis of the beam
$w(x)$	deflection of the beam
l	length of the beam
$R(x)$	residual (error) function
c_1, c_2	constants of integration
δ	Dirac delta function
$M(x)$	bending moment distribution
$Q(x)$	shear force distribution
K	dimensionless Winkler parameter
$4\lambda^4$	parameter relating beam flexural rigidity and the Winkler modulus

$w'(x) = \frac{dw}{dx}$ first derivative of $w(x)$ with respect to x.

$w''(x) = \frac{d^2w(x)}{dx^2}$ second derivative of $w(x)$ with respect to x.

Appendix 1

$$w_c = \frac{q}{2\sqrt{2}\,EI}\left(\frac{2+\sqrt{2}}{\frac{4+\pi^4}{l^4}} - \frac{2-\sqrt{2}}{\frac{81\pi^4+4}{l^4}}\right)$$
$$w_c = \frac{ql^4}{2\sqrt{2}\,EI}\left(\frac{2+\sqrt{2}}{4+\pi^4} - \frac{2-\sqrt{2}}{4+81\pi^4}\right)$$
$$w_c = \frac{ql^4}{2\sqrt{2}\,EI}\left(\frac{3.4142}{101.4091} - \frac{0.5858}{7894.1364}\right)$$
$$w_c = \frac{ql^4}{EI}\left(\frac{0.0336676 - 0.000074205}{2\sqrt{2}}\right)$$
$$w_c = 0.011877055\frac{ql^4}{EI}$$

Appendix 2

$$M_c = \frac{q\pi^2}{2\sqrt{2}\,l^2}\left(\frac{2+\sqrt{2}}{\frac{4+\pi^4}{l^4}} - \frac{9(2-\sqrt{2})}{\frac{81\pi^4+4}{l^4}}\right)$$
$$M_c = \frac{q\pi^2 l^2}{2\sqrt{2}}\left(\frac{2+\sqrt{2}}{4+\pi^4} - \frac{9(2-\sqrt{2})}{4+81\pi^4}\right)$$
$$M_c = \frac{\pi^2}{2\sqrt{2}}(0.0336676 - 0.000667845)ql^2$$
$$M_c = 0.11515ql^2$$

Appendix 3

$$Q(0) = \frac{q\pi^4}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\frac{4+\pi^4}{l^4}} + \frac{27(2-\sqrt{2})}{\frac{81\pi^4+4}{l^4}}\right)$$
$$Q(0) = \frac{q\pi^4 l}{2\sqrt{2}}\left(\frac{2+\sqrt{2}}{4+\pi^4} + \frac{27(2-\sqrt{2})}{81\pi^4+4}\right)$$
$$Q(0) = \frac{\pi^4}{2\sqrt{2}}(0.0336676 + 0.002003535)ql$$
$$Q(0) = 1.22848936ql = -Q(l)$$

Appendix 4

$$M_c = \frac{q\pi^2}{2\sqrt{2}\,l^2}\left(\frac{2+\sqrt{2}}{\frac{\pi^4+10}{l^4}} - \frac{9(2-\sqrt{2})}{\frac{81\pi^4+10}{l^4}}\right)$$
$$M_c = \frac{q\pi^2 l^2}{2\sqrt{2}}\left(\frac{2+\sqrt{2}}{\pi^4+10} - \frac{9(2-\sqrt{2})}{81\pi^4+10}\right)$$
$$M_c = 0.108589938ql^2$$
$$M_c \approx 0.10859ql^2$$

Appendix 5

$$Q(0) = \frac{q\pi^4}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\frac{\pi^4}{l^4}+\frac{10}{l^4}} + \frac{27(2-\sqrt{2})}{\frac{81\pi^4}{l^4}+\frac{10}{l^4}}\right)$$
$$Q(0) = \frac{q\pi^4}{2\sqrt{2}\,l^3}\left(\frac{2+\sqrt{2}}{\frac{\pi^4+10}{l^4}} + \frac{27(2-\sqrt{2})}{\frac{81\pi^4+10}{l^4}}\right)$$
$$Q(0) = \frac{q\pi^4 l}{2\sqrt{2}}\left(\frac{2+\sqrt{2}}{10+\pi^4} + \frac{27(2-\sqrt{2})}{81\pi^4+10}\right)$$
$$Q(0) = \frac{\pi^4}{2\sqrt{2}}(0.03178701 + 0.00200202)ql$$
$$Q(0) = 1.16367103ql$$
$$Q(0) \approx 1.16367ql$$

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